Radiation in Yang-Mills formulation of gravity and a generalized pp-wave metric

S. Başkal
Department of Physics, Middle East Technical University, Ankara 06531, Turkey

Abstract

The variational methods implemented on a quadratic Yang-Mills type Lagrangian yield two sets of equations interpreted as the field equations and the energy-momentum tensor for the gravitational field. A covariant condition is imposed on the energy-momentum tensor to represent the radiation field. A generalized pp-wave metric is found to simultaneously satisfy both the field equations and the radiation condition. The result is compared with that of Lichnérowicz.

Based on its gauge structure which was first recognized by Utiyama [1], alternative formulations of gravitation especially the ones that are derivable from quadratic Lagrangians of Yang-Mills (YM) type have received substantial interest in the literature [2, 3, 4], not merely as an academic curiosity, but essentially because when the gravitational Lagrangian is coupled to matter the renormalization problems are much less severe [5].

For massless particles like photons and neutrinos radiation presents itself, if for all observers their respective energy flows in the same direction with that of light. As to the electromagnetic field this situation is well known and the neutrino radiation in curved space is also well established [6]. It has been shown that any solution to Einstein-Weyl equations represent neutrino radiation if the energy-momentum tensor of the Weyl field can be expressed in terms of a null four vector which is collinear to its four-momentum. Therefore, there is no reason not to expect that the gauge quanta of gravitation retains the same conditions to be in a radiative state.

In this article, we begin with a YM type Lagrangian which is quadratic in the Riemann tensor and employ Palatini's variational method to derive two sets of equations whose interpretations are given as the field equations and the gravitational energy-momentum tensor. According to Palatini's method, which was first implemented by Stephenson[7] and then elaborated by Fairchild[2], the equations may be derived by independent variations of the Lagrangian for the connection and for the metric. We introduce covariant conditions on the energy-momentum tensor to represent gravitational radiation and for the solutions specialize on a generalized form of a metric which represents plane fronted waves with parallel rays (pp-wave). We compare the radiation criteria with that of Lichnérowicz.

The dynamical equations to be considered here are determined by a variational principle from the gauge-invariant action $I = \int_M L$, where M is a four dimensional spacetime endowed with a metric of $+2$ signature, and the Lagrangian L is:

$$L = \sqrt{-g}\, R^{\mu}{}_{\nu\rho\sigma}\, R^{\nu}{}_{\mu}{}^{\rho\sigma}. \tag{1}$$

Variation with respect to the connection $\delta L/\delta\Gamma^{\nu}{}_{\alpha\beta}$ gives

$$\nabla_{\mu} R^{\mu}{}_{\nu\alpha\beta} = 0, \tag{2}$$

and the Bianchi identity

$$\nabla_{[\mu} R_{\nu\sigma]\,\alpha\beta} = 0 \tag{3}$$

follows from the definition of the Riemann tensor. Variation of the action with respect to the metric is defined and interpreted as the energy-momentum tensor of the corresponding field by many authors [4, 2, 8]:

$$\delta g^{\mu\nu}\left(\frac{\delta L}{\delta g^{\mu\nu}}\right) \equiv \delta g^{\mu\nu} T_{\mu\nu}. \tag{4}$$

Here the tensor $T_{\mu\nu}$ is symmetric and takes the form

$$T_{\mu\nu} = R_{\mu\kappa}{}^{\rho}{}_{\sigma} R_{\nu}{}^{\kappa\sigma}{}_{\rho} - \frac{1}{4} g_{\mu\nu} R^{\kappa}{}_{\tau\rho\sigma} R^{\tau}{}_{\kappa}{}^{\rho\sigma}. \tag{5}$$

We define the radiation field as any solution of (2), whose energy (5) flows for all observers pointwise in the same direction with the velocity of light. That is $T_{\mu\nu}U^{\nu} \sim l^{\mu}$, where $l_{\mu}l^{\mu} = 0$ for all U^{μ} with $U^{\mu}U_{\mu} = 1$ and $\sim$ stands for proportionality. Therefore the radiation field is represented through a condition on the energy-momentum tensor as

$$T_{\mu\nu} = \rho(x) l_{\mu} l_{\nu} \tag{6}$$

where $\rho(x) > 0$ can be elucidated as the energy density. It immediately follows that the dominant energy conditions: $T_{\mu\nu}U^{\mu}U^{\nu} \geq 0$ and $T_{\mu\nu}U^{\mu}$ to be non-spacelike are satisfied. From (5) it is seen that $T_{\mu\nu}$ is traceless and therefore, the condition on l_{μ} to be an isotropic vector is already inherent in the description. The trajectories of the vector field l_{μ} are interpreted as gravitational rays.

The criterion for the existence of gravitational radiation proposed by Lichnérowicz is based on an analogy with electromagnetic radiation and imposes algebraic conditions on the Riemann tensor as [10]:

$$l_{[\mu} R_{\nu\sigma]\,\alpha\beta} = 0, \qquad l^{\mu} R_{\mu\nu\,\alpha\beta} = 0, \tag{7}$$

with $l_\mu \neq 0$ and $R_{\mu\nu\,\alpha\beta} \neq 0$. However, contracting the firs expression with $R^{\nu\sigma\kappa\tau}$ and making use of the second one, it is seen that these conditions yield to the vanishing of $T_{\mu\nu}$ and therefore the concept of energy transfer becomes ambiguous.

To gain insight into the properties of the radiation field we consider a more general form a pp-wave metric:

$$ds^2 = 2\,du\,dv + dx^2 + dy^2 + 2\,h(v,x,y,u)\,du^2. \tag{8}$$

Unlike the function in the metric that represents plane-fronted waves with parallel rays waves whose dependence is only on x, y and u [9], here the metric fuction h depends on all of the coordinates. The null vector field $l^\mu = \delta^\mu{}_1$ is subject to

$$\nabla_\mu\, l^\nu = \alpha_\mu\, l^\nu \tag{9}$$

for some vector α_μ. Contracting (9) with l^μ yields

$$l^\mu \nabla_\mu\, l^\nu = \kappa\, l^\nu \tag{10}$$

expressing that the trajectories of the vector field l^μ are null geodesics. This vector field can be chosen as $l_\mu = \partial_\mu S$ for some scalar S. It then follows that all of the optical parameters determined by the geodesic null congruence vanish. The function S is a solution of the eikonal equation $g^{\mu\nu}\,\partial_\mu S\,\partial_\nu S = 0$ and the hypersurfaces $S = \mathit{constant}$ represent the gravitational wave fronts, and are identical with the characteristics of Einstein's vacuum and Maxwell's equations [11]. Moreover, the function S obeys $g^{\mu\nu}\,\nabla_\mu \nabla_\nu S = 0$, which is the well-known wave equation.

The field equation (2) evaluated over (8) reduce to:

$$h_{vvv} = 0, \qquad h_{vvx} = 0, \qquad h_{vvy} = 0 \tag{11}$$
$$h_{vxy} = 0, \qquad h_{vxx} = 0, \qquad h_{vyy} = 0 \tag{12}$$
$$\partial_v\,(h_{xx} + h_{yy} + h_{vu}) = 0 \tag{13}$$
$$\partial_x\,(h_{xx} + h_{yy} + h_{vu}) + h_v\,h_{vx} - h_x\,h_{vv} = 0 \tag{14}$$
$$\partial_y\,(h_{xx} + h_{yy} + h_{vu}) + h_v\,h_{vy} - h_y\,h_{vv} = 0, \tag{15}$$

and the surviving components of the energy-momentum tensor are:

$$\begin{array}{llll} T_{12} = -h_{vv}^2, & T_{22} = h_{vv}^2, & T_{23} = -2\,h_{vv}\,h_{vx}, & T_{33} = h_{vv}^2, \\ & T_{34} = -2\,h_{vv}\,h_{vy}, & T_{44} = 2\,(h_{vx}^2 + h_{vy}^2 - h\,h_{vv}^2). & \end{array} \tag{16}$$

A closer examination reveals the tensorial relations

$$T_{\mu\nu} = -R\,P_{\mu\nu} + \rho\,l_\mu\,l_\nu \tag{17}$$

and

$$T_{\mu\nu} = \lambda\, g_{\mu\nu} + l_\mu k_\nu + l_\nu k_\mu, \tag{18}$$

where the scalars λ, ρ and the vector k_μ can easily be determined. Here R is the curvature scalar and $P_{\mu\nu}$ is the traceless Einstein tensor. For comparative reasons we calculate the Einstein tensor $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, which can be expressed as

$$S_{\mu\nu} = -\frac{1}{R}\, T_{\mu\nu} - \frac{1}{4}\, R\, g_{\mu\nu} + \frac{\rho}{R}\, l_\mu l_\nu. \tag{19}$$

and observe that for this particular metric the right hand side is quite different from all known forms of energy appearing in Einstein's equations.

For the metric in (8) to represent a radiation field, we first let $h_{vv} = 0$ with $\rho > 0$, where

$$\rho = 2(h_{vx}^2 + h_{vy}^2). \tag{20}$$

Therefore, h is of the form $h = B(x, y, u)\, v + C(x, y, u)$, where B and C are independent of v, and B is not a function of u only. Then, as $l_\mu = \delta_\mu^4$, the gravitational energy tensor in (5) now takes the appropriate form (6). Field equations impose more conditions on the functions B and C. From (12) it is seen that $B = \alpha(u)x + \beta(u)y + \gamma(u)$ and so (14) and (15) reduce to: $F_x = -B\, B_x$ and $F_y = -B\, B_y$, where $F = C_{xx} + C_{yy} + B_u$. These two can be written as:

$$\alpha\, F_x - \beta\, F_y = 0. \tag{21}$$

The characteristic system associated with this equation is:

$$\frac{dy}{\alpha} = \frac{dx}{-\beta} = \frac{du}{0} = \frac{df}{0}. \tag{22}$$

It admits the first integrals u, $\alpha\, x + \beta\, y + \gamma$, F, and hence F is of the form $F = F(B, u)$ and C is now any solution of $C_{xx} + C_{yy} = F - B_u$. Therefore, we have found the form of the metric which simultaneously satisfies the field equations and the radiation condition on $T_{\mu\nu}$. With this solution, the scalar ρ in (20) remains constant along the geodesic null congruence and this leads to the conservation of the gravitational energy tensor.

It is worthwhile to study the algebraic properties of the conformal Weyl tensor with respect to the principal null vector l_μ. With $l^\mu = \delta^\mu{}_1$ the following characterization for different Petrov types are obtained:

$$\begin{array}{llll}
R = 0, & h_{vx}^2 + h_{vy}^2 = 0, & h_{xy}^2 + (h_{xx} - h_{yy})^2 = 0 & \Leftrightarrow \ \text{type O}, \\
R = 0, & h_{vx}^2 + h_{vy}^2 = 0, & h_{xy}^2 + (h_{xx} - h_{yy})^2 \neq 0 & \Leftrightarrow \ \text{type N}, \\
R = 0, & h_{vx}^2 + h_{vy}^2 \neq 0, & & \Leftrightarrow \ \text{type III}, \\
R \neq 0, & & & \Leftrightarrow \ \text{type II or D}.
\end{array} \tag{23}$$

We have presented a covariant formulation of gravitational radiation based on the energy tensor of the gavitational field derived from a YM type Lagrangian. As a specific example for the illustration of a radiation field, a space-time metric admitting plane waves is considered. The plane wave is strictly parallel (i.e., α in (9) is zero) if and only if h is independent of v. Then this metric becomes identical with the one corresponding to pp-waves, which represents radiation in the sense of Lichnérowicz. However, we note that in this case, since $T_{\mu\nu} = 0$, the concept of energy transfer becomes ambiguous. The algebraic properties of the Weyl tensor are studied. We also show that this metric is never algebraically general, and that radiation corresponds to type III.

I would like to thank to F. Öktem for suggesting the radiation problem and for valuable discussions, and to T. Dereli for bringing many of the relevant papers into my attention.

References

[1] Utiyama, R.,*Phys.Rev.*, **101** (1956) 1597.

[2] Fairchild Jr., E. E. *Phys.Rev. D*, **16** (1977) 2438;

[3] Benn, I. M., Dereli, T., Tucker, R. W., *Gen. Rel. Grav.* **13** (1981) 581.

[4] Camenzind, M. *J. Math. Phys.*, **16** (1975) 1023; Mielke, E. W. *Gen. Rel. Grav.*, **13** (1981) 677.

[5] Stelle, K. S., *Phys. Rev. D*, **10** (1977) 3337.

[6] Audretsch, J. and Graf, W., *Commun. Math. Phys.*, **19** (1970) 315.

[7] Stephenson, G. *Nuovo. Cim.*, **9** (1958) 263.

[8] Başkal, S. and Dereli, T., *J. Phys. G*, **19** (1993) 477.

[9] Ehlers, J., Kundt, W., in *Gravitation: an Introduction to Current Research*, ed. Witten L. (New York, 1962); Adamowicz, A. *Gen. Rel. and Grav.* **12** (1980) 677.

[10] Lichnérowicz, A., in *Colloques Internationaux du CNRS*, Paris (1962) p. 93.

[11] Lichnérowicz, A., *Théories Relativistes de la Gravitation et de l'Electromagnétisme*, Masson, Paris (1955) p. 33.